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$SU(2) \times U(1)$ Gauge Gravity

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We propose a Lorentz-covariant Yang–Mills "spin-gauge" theory, where the function-valued Pauli matrices play the role of a nonscalar Higgs field. As symmetry group we choose $SU(2) \times U(1)$ of the 2-spinors describing particle/antiparticle states. After symmetry breaking, a nonscalar Lorentz-covariant Higgs-field gravity appears, which can be interpreted within a classical limit as Einstein's metrical theory of gravity, where we restrict ourselves in a first step to its linearized version.

1. INTRODUCTION

In a previous paper (Dehnen and Hitzer, 1994) we proposed a unitary gauge theory of gravity in view of the possibility of quantization of gravity and its unification with the other physical interactions. In this theory, where the subgroup $SU(2) \times U(1)$ of the unitary transformations of Dirac's γ-matrices between their different representations [internal spin group (see also Drechsler, 1988; Bade and Jehle, 1953; cf. also Laporte and Uhlenbeck, 1931; Barut and McEwan, 1984)] is gauged, the γ-matrices become function valued. Because the γ-matrices can be understood as the square root of the metric, our gauge group is the unitary group belonging to the square root of the metric. Taking the function-valued γ-matrices as true field variables with a Higgs–Lagrange density, and this because also the γ-matrices possess a nontrivial ground state, namely the usual constant standard representations, we obtained a unitary spin-gauge theory with Dirac's γ-matrices as Higgs fields. After spontaneous symmetry breaking, a nonscalar Higgs-gravity appears, which can be connected in a classical limit with Einstein's gravity, where we restricted ourselves in the first step for reasons of simplicity to the linear theory.

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The essential points are the following: The theory is from the beginning only Lorentz covariant. After symmetry breaking, the action of the excited γ-Higgs field on the fermions in the Minkowski space-time can be reinterpreted as if there existed non-Euclidean space-time connections and a non-Euclidean metric (effective metric), in which the fermions move freely; then the deviation from the Minkowski space-time describes classical gravity. Simultaneously the gravitational constant is produced only by the symmetry breaking whereby the gauge bosons get masses of the order of the Planck mass and can therefore be neglected in the low energy limit; but in the high-energy limit ($\simeq 10^{19}$ GeV) an additional "strong" gravitational interaction exists.

However, we found also a richer space-time geometrical structure than only a Riemannian one. We got besides an effective metric also an effective nonmetricity, whereas an effective torsion did not appear.

The aim of the present paper is to avoid nonmetricity and torsion, which is indeed possible by changing the Lagrange density. We develop first a quantum mechanical description of the gravitational interaction between fermionic elementary particles and arrive subsequently in the linearized version of the classical limit exactly at Einstein's linearized theory. However, instead of starting from Dirac's 4-spinor formalism, it is more appropriate to begin with the 2-spinor description of massless spin-1/2 fermions, because then the gauge group $SU(2) \times U(1)$ is irreducible. In consequence of this several considerations become much clearer than in the foregoing paper, especially the symmetry breaking and the transition to the classical macroscopic limit. Moreover, there is a further essential reason for starting with the 2-spinors. There exist strong hints that also for antiparticles the weak equivalence principle $\overline{m}_g \equiv \overline{m}_i$ $(\equiv m_i)$ is valid (Nieto and Goldman, 1991; see also Morrison, 1958; Ebner and Dehnen, 1993). Then particle and antiparticle are indistinguishable with respect to gravity. Therefore it is logical on the quantum mechanical level to combine particle and antiparticle as a particle doublet, on which the $SU(2) \times U(1)$ group acts. This procedure, which is exactly the same as that of electroweak and strong interactions or their unification, is possible by choosing the 2-spinors following from the chiral decomposition of Dirac's theory. For these reasons we think that the gauge group in question is indeed that of microscopic gravity, from which Einstein's macroscopic gravity follows in the classical limit as an effective field.

2. THE BASIC CONCEPT

Starting from the Lagrange density of massless spin-1/2 particles within the 4-spinor formalism ($\hbar = 1$, $c = 1$)

$$\mathcal{L}_M = \frac{i}{2}\,\bar{\psi}\gamma^\mu\partial_\mu\psi + \text{h.c.} \tag{2.1}$$

with the anticommutator relation

$$\gamma^{(\mu}\gamma^{\nu)} = \eta^{\mu\nu}\mathbf{1} \tag{2.1a}$$

[$\eta^{\mu\nu} = \eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, the Minkowski metric], we go over to a 2-spinor description using the chiral decomposition:

$$\psi = \begin{pmatrix} \chi_R \\ \varphi_L \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma_L^\mu \\ \sigma_R^\mu & 0 \end{pmatrix} \tag{2.2}$$

where

$$\sigma_L^\mu = (\mathbf{1}, -\sigma^i), \qquad \sigma_R^\mu = (\mathbf{1}, \sigma^i) \tag{2.2a}$$

(σ^i, $i = 1, 2, 3$ are Pauli matrices). Here the 2-spinor χ_R represents the right-handed particle and left-handed antiparticle states and φ_L the left-handed particle and right-handed antiparticle states. Inserting (2.2) and (2.2a) into (2.1) and (2.1a), we find the matter Lagrange density

$$\mathcal{L}_M = \frac{i}{2}\,(\chi_R^\dagger\sigma_R^\mu\partial_\mu\chi_R + \varphi_L^\dagger\sigma_L^\mu\partial_\mu\varphi_L) + \text{h.c.} \tag{2.3}$$

and the "anticommutator" relations

$$\sigma_L^{(\mu}\sigma_R^{\nu)} = \eta^{\mu\nu}\mathbf{1}, \qquad \sigma_R^{(\mu}\sigma_L^{\nu)} = \eta^{\mu\nu}\mathbf{1} \tag{2.4}$$

The Lagrange density (2.3) and the relations (2.4) are invariant or covariant under the global $SU(2) \times U(1)$ transformations

$$\begin{aligned} \chi_R' &= U\chi_R, & \varphi_L' &= U\varphi_L \\ \sigma_R^{\mu\prime} &= U\sigma_R^\mu U^{-1}, & \sigma_L^{\mu\prime} &= U\sigma_L^\mu U^{-1} \end{aligned} \tag{2.5}$$

with

$$U = e^{i\lambda_a\tau^a}, \qquad \tau^a = \frac{1}{2}\,\sigma^a, \qquad \sigma^a = (\mathbf{1}, \sigma^i)$$

$$(a = 0, 1, 2, 3; \qquad i = 1, 2, 3) \tag{2.5a}$$

[σ^i are Pauli matrices as generators of the transformation group $SU(2)$], where $\lambda_a = \text{const}$ (real valued) and the generators τ^a satisfy the commutator relation

$$[\tau^a, \tau^b] = i\epsilon^{ab}{}_c\tau^c \tag{2.5b}$$

($\epsilon^{ab}{}_c$ is the Levi-Civita symbol with the additional property to be zero if a, b, or c is zero).

Now we gauge this group by setting $\lambda_a = \lambda_a(x^\mu)$ (real-valued functions). Then the two states of χ_R and of φ_L, i.e., the (right- and left-handed) particles–antiparticles, which are mixed by this gauge group according to (2.5), are indistinguishable with respect to the resulting interaction (cf. Introduction), whereby the lepton and baryon number conservation is violated because of the possibility of particle–antiparticle transitions (see below). The invariance of the Lagrange density (2.3) is guaranteed in future by substituting the usual partial derivative by the covariant one[2]:

$$D_\mu = \partial_\mu + ig\omega_\mu$$

$$\omega'_\mu = U\omega_\mu U^{-1} + \frac{i}{g} U_{|\mu} U^{-1} \tag{2.6}$$

(g is the dimensionless gauge-coupling constant). The real-valued gauge potentials $\omega_{\mu a}$ are defined by

$$\omega_\mu = \omega_{\mu a}\tau^a \tag{2.6a}$$

However, simultaneously the σ^μ_R and σ^μ_L matrices become function valued because of the transformation law (2.5). We denote these function-valued matrices $\tilde{\sigma}^\mu_R$ and $\tilde{\sigma}^\mu_L$, respectively (bear this notation in mind!); they are Hermitian and therefore in the adjoint representation.

Furthermore, the Lagrange density (2.3) must be supplemented by two parts: first by a part for the gauge potentials $\omega_{\mu a}$ and second by a part for the $\tilde{\sigma}^\mu_R$, $\tilde{\sigma}^\mu_L$ matrix functions. We choose for the latter ones a Higgs-field Lagrange density, because $\tilde{\sigma}^\mu_R$ and $\tilde{\sigma}^\mu_L$ possess a natural nontrivial groundstate given by (2.2a). Relations (2.4) then only apply to the ground states. In general the function-valued matrices $\tilde{\sigma}^\mu_R$ and $\tilde{\sigma}^\mu_L$ later on will define an *effective* function-valued (non-Euclidean) metric $g^{\mu\nu}$.

Thus the total Lagrange density consists of three minimally coupled Lorentz- and gauge-invariant real-valued parts:

$$\mathcal{L} = \mathcal{L}_M(\psi) + \mathcal{L}_F(\omega) + \mathcal{L}_H(\tilde{\sigma}) \tag{2.7}$$

Beginning with the third brand new part, $\mathcal{L}_H(\tilde{\sigma})$ belongs to the $\tilde{\sigma}^\mu_{R,L}$ Higgs fields and we choose for this

$$\begin{aligned} \mathcal{L}_H(\tilde{\sigma}) = {} & \operatorname{tr}(D_\alpha\tilde{\sigma}_{\mu R})(D^\alpha\tilde{\sigma}^\mu_L) - \operatorname{tr}(D_\alpha\tilde{\sigma}_{\mu R})(D^\mu\tilde{\sigma}^\alpha_L) \\ & - \operatorname{tr}(D_\alpha\tilde{\sigma}^\alpha_R)(D_\beta\tilde{\sigma}^\beta_L) - V(\tilde{\sigma}) \\ & - k[\varphi^\dagger_L\tilde{\sigma}^\mu_L\tilde{\sigma}_{\mu R}\chi_R + \chi^\dagger_R\tilde{\sigma}^\mu_R\tilde{\sigma}_{\mu L}\varphi_L] \end{aligned} \tag{2.8}$$

where the last term is a Yukawa coupling term (k is a dimensionless coupling

[2] $|\mu$ denotes the partial derivative with respect to the coordinate x^μ.

constant) for generating the mass of the fermions by the $\tilde{\sigma}^{\mu}_{R,L}$ Higgs fields after spontaneous symmetry breaking.[3] The Higgs potential $V(\tilde{\sigma})$ takes the form

$$V(\tilde{\sigma}) = \mu^2 \operatorname{tr}(\tilde{\sigma}^{\mu}_{L}\tilde{\sigma}_{\mu R}) + \frac{\lambda}{12}(\operatorname{tr} \tilde{\sigma}^{\mu}_{L}\tilde{\sigma}_{\mu R})^2 \tag{2.8a}$$

($\lambda > 0$ and $\mu^2 < 0$ are real-valued constants; λ is dimensionless, and μ^2 has the dimension of a mass square). In the kinetic part of (2.8) all possible different combinations between the derivatives of $\tilde{\sigma}^{\mu}_{R}$ and $\tilde{\sigma}^{\mu}_{L}$ are taken into account.

The second term on the right-hand side of (2.7) is that of the gauge potentials $\omega_{\mu a}$ and has the usual form:

$$\mathcal{L}_F(\omega) = -\frac{1}{16\pi} F_{\mu\nu a} F^{\mu\nu}_{b} s^{ab}$$

where s^{ab} is the group metric of $SU(2) \times U(1)$ and can be taken as δ^{ab}. The gauge-field strengths are defined in the usual manner by

$$\mathcal{F}_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] = F_{\mu\nu a}\tau^a \tag{2.9a}$$

with

$$F_{\mu\nu a} = \omega_{\nu a|\mu} - \omega_{\mu a|\nu} - g\epsilon_a{}^{bc}\omega_{\mu b}\omega_{\nu c} \tag{2.9b}$$

The first term in the Lagrange density in (2.7) concerns the fermionic matter fields and is given by the gauge-invariant modification of (2.3):

$$\begin{aligned}\mathcal{L}_M(\psi) = \frac{i}{2}\{&\chi^{\dagger}_{R}\tilde{\sigma}^{\mu}_{R}D_\mu\chi_R - (D_\mu\chi_R)^{\dagger}\tilde{\sigma}^{\mu}_{R}\chi_R \\ &+\varphi^{\dagger}_{L}\tilde{\sigma}^{\mu}_{L}D_\mu\varphi_L - (D_\mu\varphi_L)^{\dagger}\tilde{\sigma}^{\mu}_{L}\varphi_L\}\end{aligned} \tag{2.10}$$

We note that the total Lagrangian (2.7) contains no dimensional parameters except μ^2 in the Higgs potential (2.8a), which has the dimension of a mass square. Because in the following by the symmetry breaking only one sort of mass is generated, the weak equivalence principle for particles and antiparticles is guaranteed from the very beginning in the general form $m_i \equiv m_g \equiv \overline{m}_i \equiv \overline{m}_g \equiv m$; see (3.5).

The field equations following from the action principle associated with (2.7) are the generalized 2-spinor equations

[3] It is worthwhile to note that this Yukawa coupling term is necessary for arriving at Einstein's theory in the classical limit.

$$i\tilde{\sigma}^{\mu}{}_{R} D_{\mu} \chi_{R} + \frac{i}{2} (D_{\mu} \tilde{\sigma}^{\mu}_{R}) \chi_{R} - k \tilde{\sigma}^{\mu}_{R} \tilde{\sigma}_{\mu L} \varphi_{L} = 0 \tag{2.11a}$$

$$i\tilde{\sigma}^{\mu}_{L} D_{\mu} \varphi_{L} + \frac{i}{2} (D_{\mu} \tilde{\sigma}^{\mu}_{L}) \varphi_{L} - k \tilde{\sigma}^{\mu}_{L} \tilde{\sigma}_{\mu R} \chi_{R} = 0 \tag{2.11b}$$

as well as their adjoint equations, and the inhomogeneous Yang–Mills equations

$$\partial_{\nu} F_{a}^{\nu\mu} + g \epsilon_{a}{}^{bc} F_{b}^{\nu\mu} \omega_{\nu c} = 4\pi j_{a}^{\mu} \tag{2.12}$$

with the gauge currents

$$j_{a}^{\mu} = j_{a}^{\mu}(\psi) + j_{a}^{\mu}(\tilde{\sigma}) \tag{2.12a}$$

consisting of a real-valued matter part

$$j_{a}^{\mu}(\psi) = \frac{g}{2} [\chi_{R}^{\dagger} \{\tilde{\sigma}^{\mu}_{R}, \tau_{a}\} \chi_{R} + \varphi_{L}^{\dagger} \{\tilde{\sigma}^{\mu}_{L}, \tau_{a}\} \varphi_{L}] \tag{2.12b}$$

and a real-valued $\tilde{\sigma}$-Higgs-field part

$$\begin{aligned} j_{a}^{\mu}(\tilde{\sigma}) = ig \operatorname{tr}\{ & [\tilde{\sigma}_{\alpha R}, \tau_{a}] D^{\mu} \tilde{\sigma}^{\alpha}_{L} + [\tilde{\sigma}_{\alpha L}, \tau_{a}] D^{\mu} \tilde{\sigma}^{\alpha}_{R} \\ & - [\tilde{\sigma}_{\alpha R}, \tau_{a}] D^{\alpha} \tilde{\sigma}^{\mu}_{L} - [\tilde{\sigma}_{\alpha L}, \tau_{a}] D^{\alpha} \tilde{\sigma}^{\mu}_{R} \\ & - [\tilde{\sigma}^{\mu}_{R}, \tau_{a}] D_{\alpha} \tilde{\sigma}^{\alpha}_{L} - [\tilde{\sigma}^{\mu}_{L}, \tau_{a}] D_{\alpha} \tilde{\sigma}^{\alpha}_{R} \} \end{aligned} \tag{2.12c}$$

Finally we get the Higgs-field equations for $\tilde{\sigma}^{\mu}_{R}$ and $\tilde{\sigma}^{\mu}_{L}$, respectively,

$$\begin{aligned} & (D_{\alpha} D^{\alpha} \tilde{\sigma}^{\mu}_{R})_{A}{}^{B} - (D_{\alpha} D^{\mu} \tilde{\sigma}^{\alpha}_{R})_{A}{}^{B} - (D^{\mu} D_{\alpha} \tilde{\sigma}^{\alpha}_{R})_{A}{}^{B} \\ & \qquad + \left[\mu^{2} + \frac{\lambda}{6} \operatorname{tr}(\tilde{\sigma}^{\alpha}_{L} \tilde{\sigma}_{\alpha R}) \right] \tilde{\sigma}^{\mu}_{RA}{}^{B} \\ & = \frac{i}{2} [\varphi_{L}^{\dagger B} (D^{\mu} \varphi_{L})_{A} - (D^{\mu} \varphi_{L})^{\dagger B} \varphi_{LA}] \\ & \quad - k[\varphi_{L}^{\dagger B} (\tilde{\sigma}^{\mu}_{R} \chi_{R})_{A} + (\chi_{R}^{\dagger} \tilde{\sigma}^{\mu}_{R})^{B} \varphi_{LA}] \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} & (D_{\alpha} D^{\alpha} \tilde{\sigma}^{\mu}_{R})_{A}{}^{B} - (D_{\alpha} D^{\mu} \tilde{\sigma}^{\alpha}_{L})_{A}{}^{B} - (D^{\mu} D_{\alpha} \tilde{\sigma}^{\alpha}_{L})_{A}{}^{B} \\ & \qquad + \left[\mu^{2} + \frac{\lambda}{6} \operatorname{tr} \tilde{\sigma}^{\alpha}_{L} \tilde{\sigma}_{\alpha R} \right] \tilde{\sigma}^{\mu}_{LA}{}^{B} \\ & = \frac{i}{2} [\chi_{R}^{\dagger B} (D^{\mu} \chi_{R})_{A} - (D^{\mu} \chi_{R})^{\dagger B} \chi_{RA})] \\ & \quad - k[\chi_{R}^{\dagger B} (\tilde{\sigma}^{\mu}_{L} \varphi_{L})_{A}] + (\varphi_{L}^{\dagger} \tilde{\sigma}^{\mu}_{L})^{B} \chi_{RA}] \end{aligned} \tag{2.14}$$

Here the lower capital Latin index A and the upper index B denote the contragradiently transformed rows and columns of the 2-spinorial quantities, respectively. The homogeneous Yang–Mills equation following from the Jacobi identity reads

$$\partial_{[\mu}F_{\nu\lambda]a} + g\epsilon_a{}^{bc}\omega_{b[\mu}F_{\nu\lambda]c} = 0 \tag{2.15}$$

The right-hand sides of (2.13) and (2.14) are Hermitian and therefore $\tilde{\sigma}_R^\mu$ and $\tilde{\sigma}_L^\mu$ remain also Hermitian in consequence of the field equations.

Finally we note the conservation laws in consequence of the invariance structure of the Lagrangian, valid modulo the field equations. First, from (2.12) the gauge-current conservation follows immediately:

$$\partial_\mu\left(j_a^\mu + \frac{g}{4\pi}\,\epsilon_a{}^{bc}F_b^{\mu\nu}\omega_{\nu c}\right) = 0 \tag{2.16}$$

Second, the energy-momentum law takes the form

$$\partial_\nu T_\mu{}^\nu = 0 \tag{2.17}$$

where $T_\mu{}^\nu$ is the gauge-invariant canonical energy-momentum tensor consisting of three real-valued parts corresponding to (2.7):

$$T_\mu{}^\nu = T_\mu{}^\nu(\psi) + T_\mu{}^\nu(\omega) + T_\mu{}^\nu(\tilde{\sigma}) \tag{2.18}$$

Here $T_\mu{}^\nu(\psi)$ has a right- and a left-handed part:

$$T_\mu{}^\nu(\psi) = T_\mu{}^\nu(\chi_R) + T_\mu{}^\nu(\varphi_L) \tag{2.19}$$

with [modulo the field equations (2.11)]

$$T_\mu{}^\nu(\chi_R) = \frac{i}{2}\,[\chi_R^\dagger\tilde{\sigma}_R^\nu D_\mu\chi_R - (D_\mu\chi_R)^\dagger\tilde{\sigma}_R^\nu\chi_R] \tag{2.19a}$$

and

$$T_\mu{}^\nu(\varphi_L) = \frac{i}{2}\,[\varphi_L^\dagger\tilde{\sigma}_L^\nu D_\mu\varphi_L - (D_\mu\varphi_L)^\dagger\tilde{\sigma}_L^\nu\varphi_L] \tag{2.19b}$$

The second term on the right-hand side of (2.18) is given as usual by

$$T_\mu{}^\nu(\omega) = -\frac{1}{4\pi}\left[F^a_{\mu\alpha}F_a^{\nu\alpha} - \frac{1}{4}\,\delta_\mu^\nu F^a_{\alpha\beta}F_a^{\alpha\beta}\right] \tag{2.20}$$

and the last term possesses the form

$$
\begin{aligned}
T_\mu{}^\nu(\tilde\sigma) = {} & \mathrm{tr}[(D^\nu\tilde\sigma^\alpha_L)(D_\mu\tilde\sigma_{\alpha R}) - (D^\alpha\tilde\sigma^\nu_L)(D_\mu\tilde\sigma_{\alpha R}) \quad - (D_\alpha\tilde\sigma^\alpha_L)(D_\mu\tilde\sigma^\nu_R) \\
& + (D^\nu\tilde\sigma^\alpha_R)(D_\mu\tilde\sigma_{\alpha L}) \quad - (D^\alpha\tilde\sigma^\nu_R)D_\mu\tilde\sigma_{\alpha L}) - (D_\alpha\tilde\sigma^\alpha_R)(D_\mu\tilde\sigma^\nu_L)] \qquad (2.21) \\
& -\delta^\nu_\mu\Big\{\mathrm{tr}[(D_\alpha\tilde\sigma_{\beta R})(D^\alpha\tilde\sigma^\beta_L) - (D_\alpha\tilde\sigma_{\beta R})(D^\beta\tilde\sigma^\alpha_L) - (D_\alpha\tilde\sigma^\alpha_R)(D_\beta\tilde\sigma^\beta_L)] \\
& \qquad -\left[\mu^2\,\mathrm{tr}(\tilde\sigma^\alpha_L\tilde\sigma_{\alpha R}) + \frac{\lambda}{12}(\mathrm{tr}\,\tilde\sigma^\alpha_L\tilde\sigma_{\alpha R})^2\right]\Big\}
\end{aligned}
$$

By insertion of (2.19)–(2.21) into (2.18) one obtains from (2.17) the momentum law for fermions. After substituting the covariant D'Alembertian of $\tilde\sigma^\mu_R$ and $\tilde\sigma^\mu_L$ by the field equations (2.13) and (2.14) one finds, using the Yang–Mills equations (2.12) and (2.15)

$$
\begin{aligned}
\partial_\nu T_\mu{}^\nu(\psi) = {} & -\frac{i}{2}\,[\varphi^\dagger_L(D_\mu\tilde\sigma^\alpha_L)D_\alpha\varphi_L - (D_\alpha\varphi_L)^\dagger(D_\mu\tilde\sigma^\alpha_L)\varphi_L \\
& + \chi^\dagger_R(D_\mu\tilde\sigma^\alpha_R)D_\alpha\chi_R - (D_\alpha\chi_R)^\dagger(D_\mu\tilde\sigma^\alpha_R)\chi_R] \\
& + k\{\varphi^\dagger_L[D_\mu(\tilde\sigma^\alpha_L\tilde\sigma_{\alpha R})]\chi_R + \chi^\dagger_R[D_\mu(\tilde\sigma^\alpha_R\tilde\sigma_{\alpha L})]\varphi_L\} \\
& + F_{\mu\nu a}j^{\nu a}(\psi) \qquad (2.22)
\end{aligned}
$$

On the right-hand side one recognizes the Lorentz-like forces of the gauge fields and the forces of the $\tilde\sigma^\mu_R$ and $\tilde\sigma^\mu_L$ Higgs fields. Although the field equations seem to be very complicated, there exists a very transparent structure and physical meaning.

First of all on the right-hand sides of (2.13) and (2.14) there appear the fermionic energy-momentum tensors (2.19a) and (2.19b), i.e., $T_\mu{}^\nu(\chi_R)$ and $T_\mu{}^\nu(\varphi_L)$ in their spinor-valued form as sources of the $\tilde\sigma^\mu_L$ and $\tilde\sigma^\mu_R$ fields, respectively, and these fields act back by their gradients on the fermions in their field equations (2.11) and the momentum law (2.22). The structure is exactly that of an attractive gravitational interaction with the energy-momentum tensor as source. However, we have two different gravitational fields, namely $\tilde\sigma^\mu_R$ and $\tilde\sigma^\mu_L$. Only in the classical limit, where no distinction of right- or left-handed states exists, do we get a universal gravitational interaction which can be described finally by a universal effective non-Euclidean metric.

3. SPONTANEOUS SYMMETRY BREAKING AND THE FERMIONIC AND BOSONIC MASSES

Although one can recognize the gravitational structure already in the foregoing section, the physical interpretation will be much clearer after symmetry breaking. The minimum of the energy-momentum tensor (2.18) in the absence of matter and gauge fields coincides with the minimum of the Higgs potential (2.8a) defined by

$$\operatorname{tr}\left(\overset{(0)}{\tilde{\sigma}}{}^{\mu}_{L}\, \overset{(0)}{\tilde{\sigma}}_{\mu R}\right) = -\frac{6\mu^2}{\lambda} = \frac{1}{2}\, v^2 \qquad (\mu^2 < 0) \tag{3.1}$$

Simultaneously, here all field equations (2.11) up to (2.15) are fulfilled. The ground states $\overset{(0)}{\tilde{\sigma}}{}^{\mu}_{L}$ and $\overset{(0)}{\tilde{\sigma}}_{\mu R}$ must be proportional to (2.2a). In view of (2.4) one then finds from (3.1)

$$\overset{(0)}{\tilde{\sigma}}{}^{\mu}_{L} = \frac{v}{4}\, \sigma^{\mu}_{L}, \qquad \overset{(0)}{\tilde{\sigma}}{}^{\mu}_{R} = \frac{v}{4}\, \sigma^{\mu}_{R} \tag{3.2}$$

Because $\tilde{\sigma}^{\mu}_{L}$ and $\tilde{\sigma}^{\mu}_{R}$ are Hermitian (adjoint representation), we can reduce them to the ground states as follows:

$$\tilde{\sigma}^{\mu}_{L} = h^{\mu}{}_{\nu L} \overset{(0)}{\tilde{\sigma}}{}^{\nu}_{L}, \qquad \tilde{\sigma}^{\mu}_{R} = h^{\mu}{}_{\nu R} \overset{(0)}{\tilde{\sigma}}{}^{\nu}_{R} \tag{3.3}$$

with the real-valued fields $h^{\mu}{}_{\nu L}$ and $h^{\mu}{}_{\nu R}$; they have the structure

$$h^{\mu}{}_{\nu L} = \delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu L}, \qquad h^{\mu}{}_{\nu R} = \delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu R} \tag{3.3a}$$

with the excited Higgs fields $\epsilon^{\mu}{}_{\nu L}$ and $\epsilon^{\mu}{}_{\nu R}$. In addition, we set

$$\chi_R = \frac{2}{\sqrt{v}}\, \chi_{RD}, \qquad \varphi_L = \frac{2}{\sqrt{v}}\, \varphi_{LD} \tag{3.4}$$

so that the real Dirac spinor ψ_D reads now [see (2.2)]

$$\psi_D = \begin{pmatrix} \chi_{RD} \\ \varphi_{LD} \end{pmatrix} \tag{3.4a}$$

In this way the dimension of $\tilde{\sigma}^{\mu}_{L}$, $\tilde{\sigma}^{\mu}_{R}$ is compensated.

Insertion of (3.2) and (3.4) into the Yukawa coupling term of the Lagrange density (2.8) using (2.4) leads immediately to the single fermionic mass m:

$$m = kv \tag{3.5}$$

On the other hand, the mass of the gauge bosons $\omega_{\mu a}$ results from the Higgs current (2.12c) in its lowest order; insertion of (3.2) gives

$$-\overset{(0)}{j}{}^{\mu}_{a}(\tilde{\sigma}) = \left[\frac{1}{2}\, \eta^{\mu\nu} M^{2\alpha}_{\alpha ab} - \left(M^{2\mu\nu}_{ab} + M^{2\nu\mu}_{ab} - \frac{1}{2}\, \eta^{\mu\nu} M^{2\alpha}_{\alpha ab}\right)\right] \omega^{b}_{\nu} \tag{3.6}$$

with the mass-square matrix

$$M^{2\mu\nu}_{ab} = \frac{-g^2 v^2}{16} \{\mathrm{tr}([\sigma^\mu_\mathrm{R}, \tau_a][\sigma^\nu_\mathrm{L}, \tau_b]) + \mathrm{tr}([\sigma^\mu_\mathrm{L}, \tau_a][\sigma^\nu_\mathrm{R}, \tau_b])\} \quad (3.6a)$$

The bracket in (3.6) consists of two parts symmetric with respect to μ, ν as well as to a, b. The first term represents the trace of $M^{2\mu\nu}_{ab}$ referring to μ, ν and the second one is the traceless symmetric part of it giving rise to an anisotropy of the (effective) mass of the gauge bosons. If we later identify the gravitational interaction mentioned in the foregoing section in its classical limit with Einstein's macroscopic metrical gravity, we will find [see (5.34)] $v^2 = (2\pi G)^{-1}$ (G is Newton's gravitational constant). Therefore the $SU(2)$ gauge bosons ($a = i = 1, 2, 3$) get masses of the order of the Planck mass $M_{\mathrm{Pl}} = 1/\sqrt{G}$ ($\triangleq 10^{19}$ GeV) according to (3.6a) and can be neglected in the low-energy limit. In the higher energy ranges, however, an additional "strong gravitational" interaction exists mediated by the three very massive $\omega_{\mu i}$ bosons, which violates, in view of the transition currents (2.12b), the lepton and baryon number conservation. With respect to the $SU(2)$ group only global transformations are allowed from now on. On the other hand, the $U(1)$ gauge boson ($a = 0$) remains massless in view of (3.6a) and (2.5a). Therefore the $U(1)$ gauge group represents the rest-symmetry group and can be identified with that of the weak hypercharge, so that here a unification with the electroweak interaction intrudes by setting $\omega_{\mu 0} = B_\mu$ (hypercharge boson).[4] But this will not be performed in this paper in any detail.

4. LOW-ENERGY LIMIT AND MICROSCOPIC GRAVITY

In this section we investigate the gravitational interaction between elementary fermionic particles after symmetry breaking under neglect of the very massive $\omega_{\mu i}$ boson interaction. Simultaneously the massless hypercharge boson is also neglected because it belongs to the range of electroweak interactions. Furthermore, for simplicity in a first step we linearize in the following with respect to $\epsilon^\mu{}_{\nu\mathrm{R}}$ and $\epsilon^\mu{}_{\nu\mathrm{L}}$ [see (3.3a)] under the assumptions that $|\epsilon^\mu{}_{\nu\mathrm{R}}| << 1$ and $|\epsilon^\mu{}_{\nu\mathrm{L}}| << 1$. Then the fermionic 2-spinor equations (2.11) using Section 3 take the form

$$i[\sigma^\mu_\mathrm{R} + \epsilon^\mu{}_{\nu\mathrm{R}}\sigma^\nu_\mathrm{R}]\, \partial_\mu \chi_\mathrm{RD} + \frac{i}{2}(\partial_\mu \epsilon^\mu{}_{\nu\mathrm{R}})\sigma^\nu_\mathrm{R}\chi_\mathrm{RD}$$

$$- m\left[1 + \frac{1}{4}(\epsilon_{\nu\sigma\mathrm{L}} + \epsilon_{\sigma\nu\mathrm{R}})\sigma^\nu_\mathrm{R}\sigma^\sigma_\mathrm{L}\right]\varphi_\mathrm{LD} = 0 \quad (4.1)$$

and

[4] In this context one has to split off a g_1 from the gauge coupling constant g.

$$i[\sigma_L^\mu + \epsilon^\mu{}_{\nu L}\sigma_L^\nu]\partial_\mu\varphi_{LD} + \frac{i}{2}(\partial_\mu\epsilon^\mu{}_{\nu L})\sigma_L^\nu\varphi_{LD}$$
$$- m\left[1 + \frac{1}{4}(\epsilon_{\nu\sigma R} + \epsilon_{\sigma\nu L})\sigma_L^\nu\sigma_R^\sigma\right]\chi_{RD} = 0 \tag{4.2}$$

where the right- and left-handed states possess the same mass m [see (3.5)].

Going over from a spinorial representation of the Higgs-field equations (2.13) and (2.14) to Lorentz-tensorial equations, we multiply these equations after symmetry breaking (without loss of generality) with $\sigma_{LB}^{\nu}{}^{A}$ and $\sigma_{RB}^{\nu}{}^{A}$, respectively, and obtain, using (2.4),

$$\partial_\alpha\partial^\alpha\epsilon_R^{\mu\nu} - 2\partial_\alpha\partial^\mu\epsilon_R^{\alpha\nu} - \frac{\mu^2}{4}(\epsilon_{\alpha R}^\alpha + \epsilon_{\alpha L}^\alpha)\eta^{\mu\nu}$$
$$= \frac{8}{v^2}\left\{T^{\mu\nu}(\varphi_{LD}) - \frac{m}{4}(\varphi_{LD}^\dagger\sigma_L^\nu\sigma_R^\mu\chi_{RD} + \chi_{RD}^\dagger\sigma_R^\mu\sigma_L^\nu\varphi_{LD})\right\} \tag{4.3}$$

and

$$\partial_\alpha\partial^\alpha\epsilon_L^{\mu\nu} - 2\partial_\alpha\partial^\mu\epsilon_L^{\alpha\nu} - \frac{\mu^2}{4}(\epsilon_{\alpha R}^\alpha + \epsilon_{\alpha L}^\alpha)\eta^{\mu\nu}$$
$$= \frac{8}{v^2}\left\{T^{\mu\nu}(\chi_{RD}) - \frac{m}{4}(\varphi_{LD}^\dagger\sigma_L^\mu\sigma_R^\nu\chi_{RD} + \chi_{RD}^\dagger\sigma_R^\nu\sigma_L^\mu\varphi_{LD})\right\} \tag{4.4}$$

Here $T^{\mu\nu}(\chi_{RD})$ and $T^{\mu\nu}(\varphi_{LD})$ are the right- and left-handed energy-momentum tensors (2.19a) and (2.19b) in their lowest order and v^{-2} plays the role of the gravitational constant [cf. (5.34)].

Here a certain cross-interaction between the right- and left-handed states exists, which is already present in the original equations (2.13) and (2.14): The energy of the right-handed states $[T^{\mu\nu}(\chi_{RD})]$ generates the gravitational fields $\epsilon_L^{\mu\nu}$ according to (4.4), which act back on the left-handed states (φ_{LD}) in view of (4.2) and vice versa. However, because of the mass terms in (4.1), (4.2) and (4.3), (4.4) this cross-interaction picture applies only in the massless case ($k = 0$) exactly. In consequence of this there exists no neutrino–neutrino interaction by the microscopic gravitational $\epsilon_L^{\mu\nu}$, $\epsilon_R^{\mu\nu}$ fields, if only left-handed neutrinos exist.

5. MACROSCOPIC GRAVITY

We neglect furthermore the gauge-boson interaction. Moreover, in the classical macroscopic limit right- and left-handed states may be equally represented, i.e.,

$$T^{\mu\nu}(\chi_{RD}) = T^{\mu\nu}(\varphi_{LD}) = \frac{1}{2}T^{\mu\nu}(\psi_D) \tag{5.1}$$

and in consequence of this, according to (4.3) and (4.4),

$$\epsilon_{\mathrm{R}}^{\mu\nu} = \epsilon_{\mathrm{L}}^{\mu\nu} = \epsilon^{\mu\nu} \tag{5.2a}$$

and in view of (3.3a)

$$h_{\mathrm{R}}^{\mu\nu} = h_{\mathrm{L}}^{\mu\nu} = h^{\mu\nu} \tag{5.2b}$$

In this and only this case we can define generalized Dirac matrices $\tilde{\gamma}^{\mu}$ by setting in view of (2.2)

$$\tilde{\gamma}^{\mu} = h^{\mu}{}_{\nu}\begin{pmatrix} 0 & \sigma_{\mathrm{L}}^{\nu} \\ \sigma_{\mathrm{R}}^{\nu} & 0 \end{pmatrix} = h^{\mu}{}_{\nu}\gamma^{\nu} \tag{5.3}$$

For this reason (5.1) is necessary in the classical limit. Then the fermionic 2-spinor equations (4.1) and (4.2) can be combined into a generalized Dirac equation for the 4-spinor ψ_{D} [see (3.4a)]:

$$i\tilde{\gamma}^{\mu}\partial_{\mu}\psi_{\mathrm{D}} + \frac{i}{2}(\partial_{\mu}\tilde{\gamma}^{\mu})\psi_{\mathrm{D}} - \frac{m}{4}\tilde{\gamma}^{\mu}\tilde{\gamma}_{\mu}\psi_{\mathrm{D}} = 0 \tag{5.4}$$

Simultaneously by addition of the Higgs-field equations (4.3) and (4.4) we obtain ($\epsilon = \epsilon_{\mu}^{\mu}$)

$$\partial_{\alpha}\partial^{\alpha}\epsilon^{\mu\nu} - 2\partial_{\alpha}\partial^{\mu}\epsilon^{\alpha\nu} - \frac{\mu^2}{2}\epsilon\eta^{\mu\nu}$$
$$= \frac{4}{v^2}\left\{T^{\mu\nu}(\psi_{\mathrm{D}}) - \frac{1}{2}T(\psi_{\mathrm{D}})\eta^{\mu\nu}\right\} \tag{5.5}$$

where in the lowest order (modulo Dirac equation)

$$T(\psi_{\mathrm{D}}) = m(\varphi_{\mathrm{LD}}^{\dagger}\chi_{\mathrm{RD}} + \chi_{\mathrm{RD}}^{\dagger}\varphi_{\mathrm{LD}}) \tag{5.5a}$$

is the trace of Dirac's canonical energy-momentum tensor $T^{\mu\nu}(\psi_{\mathrm{D}})$. Finally, the momentum law (2.22) takes in this classical limit after a longer calculation the very simple form

$$\partial_{\nu}T_{\mu}{}^{\nu}(\psi_{\mathrm{D}}) = -(\partial_{\mu}\epsilon^{\alpha}{}_{\beta})T_{\alpha}{}^{\beta}(\psi_{\mathrm{D}}) + \frac{1}{2}(\partial_{\mu}\epsilon)T(\psi_{\mathrm{D}}) \tag{5.6}$$

Here the question of a connection of (5.5) and (5.6) to Einstein's metrical theory of gravity arises. For this we have to define first an effective non-Euclidean metric.

5.1. The Effective Metric

We define the effective metric by the mass-shell condition following from the Dirac equation in the lowest WKB limit. For this we insert (5.3) into (5.4) and find (linearized in $\epsilon^{\mu\nu}$)

$$i\gamma^{\mu}\mathcal{D}_{\mu}\psi_{\mathrm{D}} - m\left(1 + \frac{1}{2}\epsilon\right)\psi_{\mathrm{D}} = 0 \tag{5.7}$$

with the generalized derivative

$$\mathcal{D}_{\mu} = \partial_{\mu} + \epsilon^{\nu}{}_{\mu}\partial_{\nu} + \frac{1}{2}(\partial_{\nu}\epsilon^{\nu}{}_{\mu}) \tag{5.7a}$$

Iteration of (5.7), elimination of spin-operator influences because of aspiring to the classical limit, and consequent linearization in $\epsilon^{\mu\nu}$ gives

$$\mathcal{D}_{\mu}\mathcal{D}^{\mu}\psi_{\mathrm{D}} + m^2(1 + \epsilon)\psi_{\mathrm{D}} + \frac{i}{2} m\gamma^{\mu}(\partial_{\mu}\epsilon)\psi_{\mathrm{D}} = 0 \tag{5.8}$$

or after insertion of (5.7a)

$$\begin{aligned} &\Big(\partial_{\mu}\partial^{\mu} + 2\epsilon^{(\nu\mu)}\partial_{\nu}\partial_{\mu} + \epsilon^{\nu\mu}{}_{|\mu}\partial_{\nu} \\ &\quad + \epsilon^{\nu\mu}{}_{|\nu}\partial_{\mu} + \frac{1}{2}\epsilon^{(\nu\mu)}{}_{|\nu|\mu}\Big)\psi_{\mathrm{D}} \\ &\quad + \frac{m^2}{\hbar^2}(1 + \epsilon)\psi_{\mathrm{D}} + \frac{i}{2}\frac{m}{\hbar}\gamma^{\mu}(\partial_{\mu}\epsilon)\psi_{\mathrm{D}} = 0 \end{aligned} \tag{5.8a}$$

where we have introduced $\hbar$ explicitly in view of the WKB method. The γ^{μ} term will vanish in the following because of (5.23), so that (5.8a) has indeed the structure of a Klein–Gordon equation.

Now we use the WKB Ansatz

$$\psi_{\mathrm{D}} = Ae^{iW/\hbar} \tag{5.9}$$

(A is a 4-spinorial amplitude, W a scalar phase function) and expand W and A with respect to $\hbar$ as follows:

$$\begin{aligned} W &= \sum_{n=0} \overset{(n)}{W}\hbar^n \\ A &= \sum_{n=0} \overset{(n)}{A}\hbar^n \end{aligned} \tag{5.9a}$$

Insertion of (5.9) and (5.9a) into (5.8a) gives in the lowest order of $\hbar$ (i.e., $\hbar^0$) the Hamilton–Jacobi equation:

$$(\eta^{\mu\nu} + 2\epsilon^{(\mu\nu)})\overset{(0)}{W}_{|\mu}\overset{(0)}{W}_{|\nu} - m^2(1 + \epsilon) = 0 \tag{5.10}$$

which is simultaneously the mass-shell condition for the canonical 4-momentum of the particle:

$$p_\mu = \overset{(0)}{W}_{|\mu} \tag{5.10a}$$

Consequently, the effective non-Euclidean metric is defined by

$$g^{\mu\nu} = \eta^{\mu\nu}(1 - \epsilon) + 2\epsilon^{(\mu\nu)} \tag{5.11a}$$

and because of[5] $g_{\mu\nu}g^{\nu\lambda} = \delta^\lambda_\mu$

$$g_{\mu\nu} = \eta_{\mu\nu}(1 + \epsilon) - 2\epsilon_{(\mu\nu)} \tag{5.11b}$$

so that equation (5.10) takes the form of the mass shell

$$g^{\mu\nu}p_\mu p_\nu - m^2 = 0 \tag{5.12}$$

We note here that such an effective general metric for describing gravity is only possible in the classical limit defined by (5.1) and (5.2).

Finally, we derive the equation of motion of the quantum particle in its classical limit following from (5.12) by the 4-gradient; one finds in view of (5.10a)

$$p_{\alpha|\mu}p_\nu g^{\mu\nu} + \frac{1}{2} g^{\mu\nu}{}_{|\alpha}p_\mu p_\nu = 0 \tag{5.13}$$

This is exactly the equation of geodesics with the effective metric $g^{\mu\nu}$ and its Christoffel symbols $\{^\mu{}_{\alpha\beta}\}$ as connection coefficients and can be written in the form[6]

$$p_{\alpha\|\mu}p_\nu g^{\mu\nu} = 0 \tag{5.13a}$$

Consequently the effective non-Euclidean space-time is a Riemannian one. On the other hand, we note that the metric (5.11a) is connected with the generalized Dirac matrices (5.3) by the anticommutator relation:

$$2g^{\mu\nu}\mathbf{1} = \{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\}(1 - \epsilon) \tag{5.14}$$

Thus, only if the trace is $\epsilon \equiv 0$ do the matrices $\tilde{\gamma}^\mu$ define a Clifford algebra on the effective metric $g^{\mu\nu}$. In the next section we shall show that $\epsilon \equiv 0$ is indeed valid.

5.2. The Gravitational Field Equations

In the next step we derive the field equations for the effective metric (5.11) starting from the Higgs equations (5.5). First we take the trace of (5.5):

[5] Note that here the indices are not lowered by $\eta_{\mu\nu}$!

[6] $\|\mu$ denotes the covariant derivative with respect to the effective metric and its Christoffel symbols.

$$\epsilon^{|\alpha}{}_{|\alpha} - 2\mu^2\epsilon - 2\epsilon^{(\alpha\beta)}{}_{|\alpha|\beta} = -\frac{4}{v^2}\, T(\psi_D) \tag{5.15}$$

and eliminate $T(\psi_D)$ in (5.5), giving

$$\epsilon^{\mu\nu|\alpha}{}_{|\alpha} - 2\epsilon^{\alpha\nu}{}_{|\alpha}{}^{|\mu} + \left(\epsilon^{(\alpha\beta)}{}_{|\alpha|\beta} - \frac{1}{2}\,\epsilon^{|\alpha}{}_{|\alpha} + \frac{\mu^2}{2}\,\epsilon\right)\eta^{\mu\nu} = \frac{4}{v^2}\, T^{\mu\nu}(\psi_D) \tag{5.16}$$

Subsequently we decompose (5.16) into its symmetric and antisymmetric part, resulting in

$$\epsilon^{(\mu\nu)|\alpha}{}_{|\alpha} - \epsilon^{(\alpha\nu)}{}_{|\alpha}{}^{|\mu} - \epsilon^{[\alpha\nu]}{}_{|\alpha}{}^{|\mu} - \epsilon^{(\alpha\mu)}{}_{|\alpha}{}^{|\nu} - \epsilon^{[\alpha\mu]}{}_{|\alpha}{}^{|\nu}$$

$$+ \left(\epsilon^{(\alpha\beta)}{}_{|\alpha|\beta} - \frac{1}{2}\,\epsilon^{|\alpha}{}_{|\alpha} + \frac{\mu^2}{2}\,\epsilon\right)\eta^{\mu\nu} = \frac{4}{v^2}\, T^{(\mu\nu)}(\psi_D) \tag{5.17}$$

and

$$\epsilon^{[\mu\nu]|\alpha}{}_{|\alpha} - \epsilon^{(\alpha\nu)}{}_{|\alpha}{}^{|\mu} - \epsilon^{[\alpha\nu]}{}_{|\alpha}{}^{|\mu} + \epsilon^{(\alpha\mu)}{}_{|\alpha}{}^{|\nu} + \epsilon^{[\alpha\mu]}{}_{|\alpha}{}^{|\nu} = \frac{4}{v^2}\, T^{[\mu\nu]}(\psi_D) \tag{5.18}$$

In the lowest order considered here ($|\epsilon^{\mu\nu}| << 1$) it follows from (5.6) that

$$T^{\mu\nu}(\psi_D)_{|\nu} = 0 \tag{5.19}$$

which decomposes automatically into

$$T^{(\mu\nu)}(\psi_D)_{|\nu} = 0 \tag{5.19a}$$

and

$$T^{[\mu\nu]}(\psi_D)_{|\nu} = 0 \tag{5.19b}$$

in consequence of the fact that in Dirac's theory also the divergence of the symmetrized canonical energy-momentum tensor vanishes.

Applying first condition (5.19a) to (5.17) gives

$$q^{\mu|\nu}{}_{|\nu} - \frac{\mu^2}{2}\,\epsilon^{|\mu} \equiv 0 \tag{5.20}$$

with the abbreviation

$$q^{\mu} = \epsilon^{[\alpha\mu]}{}_{|\alpha} + \frac{1}{2}\,\epsilon^{|\mu} \tag{5.20a}$$

Taking in (5.20) the 4-divergence with respect to x^{μ}, we get immediately

$$\Box(\epsilon^{|\mu}{}_{|\mu} - \mu^2\epsilon) \equiv 0 \tag{5.21}$$

The only solution of (5.21) without any source reads (note that ϵ vanishes asymptotically)

$$\Box\epsilon - \mu^2\epsilon \equiv 0 \tag{5.22}$$

which has also the only source-free solution

$$\epsilon \equiv 0 \tag{5.23}$$

The Higgs-mass $(-\mu^2)$ is connected with the vanishing trace alone, i.e., the massive Higgs state is not excited. Now, Equation (5.20) takes the form

$$\Box q^{\mu} \equiv 0 \tag{5.24}$$

Again the only source-free solution is (note that also q^{μ} vanishes asymptotically)[7]

$$q^{\mu} \equiv 0 \tag{5.25}$$

which results, in view of (5.20a) and (5.23), in

$$\epsilon^{[\alpha\mu]}{}_{|\alpha} \equiv 0 \tag{5.26}$$

With (5.23) and (5.26) the first condition (5.20) is fulfilled. The second condition, by applying (5.19b) to (5.18), reads with the use of (5.26)[8]

$$(\epsilon^{(\alpha\mu)}{}_{|\alpha}{}^{|\nu} - \epsilon^{(\alpha\nu)}{}_{|\alpha}{}^{|\mu})_{|\nu} \equiv 0 \tag{5.27}$$

Before we investigate condition (5.27) in more detail we compare our result with Einstein's theory of gravity. First we note that, because of (5.23), the effective metric (5.11) reads finally

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu} + 2\epsilon^{(\mu\nu)} \\ g_{\mu\nu} &= \eta_{\mu\nu} - 2\epsilon_{(\mu\nu)} \end{aligned} \tag{5.28}$$

and (5.14) yields

$$\{\tilde{\gamma}^{\mu}, \tilde{\gamma}^{\nu}\} = 2g^{\mu\nu}\mathbf{1} \tag{5.28a}$$

defining a Clifford algebra on the effective metric $g^{\mu\nu}$. Because of (5.23) and (5.26) the field equations (5.17) and (5.18) take the form

$$\epsilon^{(\mu\nu)|\alpha}{}_{|\alpha} - \epsilon^{(\alpha\nu)}{}_{|\alpha}{}^{|\mu} - \epsilon^{(\alpha\mu)}{}_{|\alpha}{}^{|\nu} + \epsilon^{(\alpha\beta)}{}_{|\alpha|\beta}\eta^{\mu\nu} = \frac{4}{v^2}T^{(\mu\nu)}(\psi_D) \tag{5.29}$$

and

[7] In view of (5.36) it may be of interest that also q^{μ} is a 4-gradient: From (5.20) it follows immediately that $\Box(q^{\mu|\nu} - q^{\nu|\mu}) \equiv 0$ with the only source-free solution $q^{\mu|\nu} - q^{\nu|\mu} \equiv 0$, so that $q^{\mu} = q^{|\mu}$. In view of (5.25), it follows that $q = \text{const} = 0$ (const $= 0$ because of the boundary condition at infinity).

[8] The appearance of gauge conditions for the Higgs fields in consequence of the conservation laws is a result of the symmetry breaking.

$$\epsilon^{[\mu\nu]|\alpha}{}_{|\alpha} + \epsilon^{(\alpha\mu)}{}_{|\alpha}{}^{|\nu} - \epsilon^{(\alpha\nu)}{}_{|\alpha}{}^{|\mu} = \frac{4}{v^2} T^{[\mu\nu]}(\psi_D) \qquad (5.30)$$

It is remarkable that because of (5.23) the Higgs mass term proportional to μ^2 in (5.17) drops out. The gravitational field is massless, i.e., it consists of the massless Goldstone states of the $\tilde{\sigma}$ Higgs field alone.

Now we compare (5.29) with Einstein's linearized field equations, setting there

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \qquad (|\gamma_{\mu\nu}| << 1) \qquad (5.31)$$

Comparison with (5.28) gives first

$$\gamma_{\mu\nu} = -2\epsilon_{(\mu\nu)} \qquad (5.31a)$$

Then the condition (5.23) means

$$\gamma^\alpha{}_\alpha \equiv \gamma \equiv 0 \Leftrightarrow \det g_{\mu\nu} \equiv g \equiv -1 \qquad (5.32)$$

Thus we have to take Einstein's field equations in the special gauge (5.32), which Einstein used already in his basic paper (Einstein, 1916) (Einstein gauge).[9] In this gauge Einstein's linearized equations are given by

$$\frac{1}{2}[\gamma_{\mu\nu}{}^{|\alpha}{}_{|\alpha} - \gamma_\nu{}^\alpha{}_{|\alpha|\mu} - \gamma_\mu{}^\alpha{}_{|\alpha|\nu} + \gamma_{\alpha\beta}{}^{|\alpha|\beta}\eta_{\mu\nu}] = -8\pi G T_{(\mu\nu)} \quad (5.33)$$

Insertion of (5.31a) shows that equation (5.33) is identical with (5.29) if we set

$$v^2 = (2\pi G)^{-1} \qquad (5.34)$$

The gravitational constant is a consequence of the symmetry breaking! In the classical limit our theory coincides in its linearized version exactly with Einstein's linearized theory of gravity.

Let us now consider the integration procedure of the field equations (5.29) and (5.30) under the conditions (5.23), (5.26), and (5.27). For this we investigate first the condition (5.27) in more detail. It means

$$\epsilon^{(\alpha\mu)}{}_{|\alpha}{}^{|\nu}{}_{|\nu} = \epsilon^{(\alpha\nu)}{}_{|\alpha|\nu}{}^{|\mu} = w^{|\mu} \qquad (5.35)$$

($w = \epsilon^{(\alpha\nu)}{}_{|\alpha|\nu}$) and therefore

$$\Box\epsilon^{(\alpha\mu)}{}_{|\alpha} = w^{|\mu} \qquad (5.35a)$$

Then it follows immediately that

$$\Box(\epsilon^{(\alpha\mu)}{}_{|\alpha}{}^{|\nu} - \epsilon^{(\alpha\nu)}{}_{|\alpha}{}^{|\mu}) \equiv 0 \qquad (5.35b)$$

[9]This gauge has the advantage that the nonlinear Einstein equations become polynomial, comparable with our nonlinear theory.

with the only totally source-free solution

$$\epsilon^{(\alpha\mu)}{}_{|\alpha}{}^{|\nu} - \epsilon^{(\alpha\nu)}{}_{|\alpha}{}^{|\mu} \equiv 0 \tag{5.35c}$$

Hence $\epsilon^{(\alpha\mu)}{}_{|\alpha}$ is a 4-gradient, i.e.,

$$\epsilon^{(\alpha\mu)}{}_{|\alpha} = f^{|\mu} \tag{5.36}$$

Herewith the condition (5.27) is fulfilled.

In consequence of the remaining gauge conditions (5.23), (5.26), and (5.36) the field equations (5.29) and (5.30) take the following final form using (5.34):

$$\epsilon^{(\mu\nu)|\alpha}{}_{|\alpha} - 2f^{|\mu|\nu} + f^{|\alpha}{}_{|\alpha}\eta^{\mu\nu} = 8\pi G T^{(\mu\nu)}(\psi_D) \tag{5.37a}$$

$$f^{|\alpha}{}_{|\alpha} = 4\pi G T(\psi_D) \tag{5.37b}$$

$$\epsilon^{[\mu\nu]|\alpha}{}_{|\alpha} = 8\pi G T^{[\mu\nu]}(\psi_D) \tag{5.37c}$$

Because of the vanishing divergences of $T^{(\mu\nu)}(\psi_D)$ and $T^{[\mu\nu]}(\psi_D)$ [see (5.19)], all gauge conditions are also consequences of the field equations (5.37), which can be integrated by retarded integrals in a straightforward manner. Equation (5.37b) is the Lorentz-invariant Poisson equation for the Lorentz-invariant generalization of the Newtonian gravitational potential, so that (up to the sign) f has this meaning. Of course, the same integration procedure is possible for Einstein's field equations (5.33) with the gauge condition (5.32). Finally, we note that for describing classical macroscopic gravity only the field equations (5.37a) and (5.37b) are necessary. Equation (5.37c) for the antisymmetric part of $\epsilon^{\mu\nu}$ does not possess a classical analog in Einstein's theory and $\epsilon^{[\mu\nu]}$ plays no role in the classical limit. However, in the complete Dirac equation (5.7) it appears, whereas it drops out in the Klein–Gordon equation (5.8a) because of (5.26). Consequently $\epsilon^{[\mu\nu]}$ couples only to spin properties, which is confirmed by its source in (5.37c) taking the form in the lowest order

$$T^{[\mu\nu]}(\psi_D) = \frac{1}{2}\,[\bar{\psi}_D\gamma^{[\mu}\sigma^{\nu]\lambda}\partial_\lambda\psi_D + (\partial_\lambda\bar{\psi}_D)\sigma^{\lambda[\mu}\gamma^{\nu]}\psi_D] \tag{5.38}$$

where $\sigma^{\mu\nu} = i\gamma^{[\mu}\gamma^{\nu]}$ is the spin operator.

5.3. The Equations of Motion

By equation (5.13a) we have already shown that the quantum particle in its classical limit moves along geodesics. Here we prove first that the momentum law (5.6) is exactly that of Einstein's theory and second that the iterated Dirac equation (5.8a) is identical with that in Einstein's theory.

Concerning the momentum law (5.6), we decompose on the right-hand side the energy-momentum tensor into its symmetric and antisymmetric part; because of (5.23) we find

$$\partial_\nu T_\mu{}^\nu(\psi_D) = -(\partial_\mu \epsilon_{(\alpha\beta)}) T^{(\alpha\beta)}(\psi_D) - (\partial_\mu \epsilon_{[\alpha\beta]}) T^{[\alpha\beta]}(\psi_D) \qquad (5.39)$$

Here it is confirmed once more that $\epsilon_{[\mu\nu]}$ couples only to spin properties, which are also its source [see (5.37c), (5.38)].[10] But because we have to neglect all spin influences within the classical limit, equation (5.39) goes over into

$$\partial_\nu T_\mu{}^\nu(\psi_D) = -(\partial_\mu \epsilon_{(\alpha\beta)}) T^{(\alpha\beta)}(\psi_D) \qquad (5.39a)$$

This equation, in which $\epsilon_{(\alpha\beta)}$ acts back on its source according to (5.37a), is identical with that in Einstein's theory for the symmetric energy-momentum tensor:

$$T_\mu{}^\nu{}_{\|\nu} = 0 \Rightarrow \partial_\nu T_\mu{}^\nu = -\begin{Bmatrix} \alpha \\ \alpha\beta \end{Bmatrix} T_\mu{}^\beta + \begin{Bmatrix} \alpha \\ \beta\mu \end{Bmatrix} T_\alpha{}^\beta \qquad (5.40)$$

if we take into account

$$g = -1 \to \begin{Bmatrix} \alpha \\ \alpha\beta \end{Bmatrix} = 0$$

$$\begin{Bmatrix} \alpha \\ \beta\alpha \end{Bmatrix} = -\epsilon^{(\alpha}{}_{\mu)|\beta} - \epsilon_{(\beta}{}^{\alpha)}{}_{|\mu} + \epsilon_{(\beta\mu)}{}^{|\alpha} \qquad (5.40a)$$

according to (5.31a) and (5.32).

Second, because of the conditions (5.23) and (5.26), the iterated Dirac equation (5.8a) takes the form of the Klein–Gordon equation:

$$\left(\partial_\mu \partial^\mu + 2\epsilon^{(\nu\mu)} \partial_\nu \partial_\mu + 2\epsilon^{(\nu\mu)}{}_{|\mu} \partial_\nu + \frac{1}{2}\, \epsilon^{(\nu\mu)}{}_{|\nu|\mu}\right)\psi_D + m^2 \psi_D = 0 \quad (5.41)$$

Using (5.28) for elimination of $\epsilon^{(\mu\nu)}$, we get

$$(\partial_\mu \partial_\nu \psi_D) g^{\mu\nu} + g^{\mu\nu}{}_{|\mu} \partial_\nu \psi_D + \frac{1}{4}\, g^{\mu\nu}{}_{|\mu|\nu} \psi_D + m^2 \psi_D = 0 \qquad (5.42)$$

Because of (R is the Ricci scalar)

$$R = g^{\mu\nu}{}_{|\mu|\nu} \qquad (g = 1) \qquad (5.42a)$$

[10]The second term on the right-hand side of (5.39) and equation (5.37c) are comparable with the torsion expressions in the Poincaré gauge theory proposed by Hehl (1973, 1974). However, an interpretation of these supplements as such of an effective torsion is not possible.

and $\{^{\alpha}_{\alpha\mu}\} = 0$ [see (5.40a)] equation (5.42) goes over into

$$(\psi_{D|\nu} g^{\mu\nu})_{\|\mu} + \frac{1}{4} R\psi_D + m^2\psi_D = 0 \tag{5.43}$$

This is in the vacuum ($R \equiv 0$) the minimally coupled general covariant Klein–Gordon equation. In the framework of the microphysics the term $\frac{1}{4}R\psi_D$ plays no role. Consequently not only on the level of classical mechanics [equation (5.40)], but also on the quantum mechanical level [equation (5.43)] our theory is in accordance with Einstein's gravitational theory within the linearized versions.

6. CONCLUSIONS

We have shown that a "spin-gauge" theory of the group $SU(2) \times U(1)$ of the 2-spinors representing particle/antiparticle results in a gravitational interaction between elementary spin-1/2 particles after symmetry breaking. The function-valued Pauli matrices are treated as Higgs fields and mediate a gravitational cross-interaction between right- and left-handed states. In the classical limit, where right- and left-handed states are equally represented, the gravitational interaction can be described by an effective metric totally in accordance with Einstein's metrical theory of gravity, where we have restricted ourselves for simplicity to the linearized version of the theories. The comparison of the nonlinear theories is under investigation.

After symmetry breaking, the $SU(2)$ gauge bosons become very massive of the order of the Planck mass and give rise to an additional "strong gravitational" interaction at very high energies ($\sim 10^{19}$ GeV) connected with particle–antiparticle transitions; this is of interest in view of the particle/antiparticle asymmetry in the universe. However, the $U(1)$ gauge boson remains massless, so that it can be identified with that of the (weak) hypercharge. Here a unification with the electroweak interaction may be possible on the basis of unitary phase gauge transformations within a high-dimensional (e.g., 4-dimensional) spin–isospin space, which decays after symmetry breaking into a spin space and an isospin space describing gravitational and electroweak interactions separately.

We hope that by such a procedure also the chiral asymmetry of the fermions with regard to the weak interaction, which is already present in the $SU(5)$ GUT, can be explained as a consequence of the symmetry breaking mentioned above. Second, the theory, as it stands, describes the gravitational interaction between fermions only. Within a complete gravitational theory the interaction with all bosons must be included, which may also require a unification with the electroweak interaction. We present this in a subsequent paper.

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